Abstract Algebra HW 6 Bonus 2

LRYP

Problem

Define the rank of a group G, denoted $\text{rank}(G)$, as the smallest cardinality of a generating set for G. For example:

- $\text{rank}(1) = 0$ where 1 is the trivial group.
- $\text{rank}(\mathbb{Z}_8) = 1$ since $\mathbb{Z}_8 = \langle x \mid x^8 = 1 \rangle$.
- $\text{rank}(D_8) = 2$ since $D_8 = \langle r, s \mid r^4 = s^2 = 1, rs = sr^{-1} \rangle$.
- $\text{rank}(E_8) = 3$ since $E_8 = \langle a, b, c \mid a^2 = b^2 = c^2 = 1, ab = ba, ac = ca, bc = cb \rangle$.

Here we have a false proposition:

> *For any group G and subgroup $H \leq G$,* $\text{rank}(H) \leq \text{rank}(G)$.

Answer the following questions:

(1) Find a counterexample, where G and H are finite groups.

(2) Find a counterexample, where G and H are infinite but finitely generated groups.

(3) Find the smallest order n, such that there exists a counterexample with $|G| = n$.

Solution

(1) By Cayley's Theorem, any group of order n is isomorphic to a subgroup of S_n. So S_8 has some subgroup isomorphic to E_8. $\text{rank}(S_8) = 2$ since $S_8 = \langle (12), (12345678) \rangle$, but $\text{rank}(E_8) = 3$.

(2) Let $G = F_2 = \langle a, b \mid \ \rangle$, the free group of rank 2. Let $H = \langle a^2, ab, b^2 \rangle \leq G$, we claim that $H \cong F_3$, thus $\text{rank}(G) = 2 < \text{rank}(H) = 3$.

Denote $x = a^2, y = ab, z = b^2$. It is sufficient to prove that, any nonempty reduced word on $\{x, y, z\}$ does not correspond to the empty word in G.

For any nonempty reduced word w on $\{x, y, z\}$, let $f(w)$ be its corresponding reduced word in G, and $|f(w)|$ be the reduced length of $f(w)$. Denote the last symbol of w be s. We state that: $\forall t \in \{x, y, z, x^{-1}, y^{-1}, z^{-1}\} \setminus \{s^{-1}\}$, $|f(w)| \leq |f(wt)|$, i.e. $f(w)$ cannot end with $f(t^{-1})$. We prove it by claiming:

- If $s = x$, then $f(w)$ can only end with aa, $b^{-1}a$ or ba;

- If $s = y$, then $f(w)$ can only end with ab or $a^{-1}b$;
- If $s = z$, then $f(w)$ can only end with bb;
- If $s = x^{-1}$, then $f(w)$ can only end with $a^{-1}a^{-1}$;
- If $s = y^{-1}$, then $f(w)$ can only end with $b^{-1}a^{-1}$ or ba^{-1};
- If $s = z^{-1}$, then $f(w)$ can only end with $b^{-1}b^{-1}$, ab^{-1} or $a^{-1}b^{-1}$;

which can be checked by going through the possibility presented by the claim itself, and at the same time, $|f(w)| \le |f(wt)|$ is automatically proven.

Now, if w is a single symbol, $|f(w)| = 2$, so for any nonempty reduced word w, $f(w)$ is not the empty word of G. Thus $H \cong \langle x, y, z \mid \rangle = F_3$.

P.S. The possibility of $(s, f(w))$ is generated by iteratively running on the DFA of the binary operation of the words, so drawing up the DFA explicitly is also sufficient.

Also, there exists advanced proving strategy from algebraic geometry.

(3) If $\operatorname{rank}(G) = 1$, then G is a cyclic group, which has no noncyclic subgroup. So we should find $\operatorname{rank}(G) = 2$ and $\operatorname{rank}(H) = 3$. The minimal order of a group of rank 3 is 8—E_8. Thus we should consider G of order 16, 24, $\cdots$.

If $|G| = 16$, then $E_8 \cong H \trianglelefteq G$ since any subgroup of index 2 must be normal. Suppose H is generated by a, b, c. Take any $d \in G \setminus H$, since $\bar{d} \cdot \bar{d} = \bar{1}$ in $G/H \cong \mathbb{Z}_2$, $d^2 \in H$. We consider these relations: $d^2 = a$, $dbd^{-1} = c$. The resulting group can be generated by b and d. So we have the group

$$\begin{aligned} G &= E_8 \rtimes \mathbb{Z}_2 \\ &\cong \langle a, b, c, d \mid a^2 = b^2 = c^2 = 1, ab = ba, ac = ca, bc = cb, d^2 = a, dbd^{-1} = c \rangle \\ &\cong \langle x, y \mid x^2 = y^4 = 1, xy^2 = y^2x, xyxy^{-1} = yxy^{-1}x \rangle, \end{aligned}$$

which satisfies $\operatorname{rank}(G) = 2$, and has a subgroup of rank 3. (P.S. Here the semidirect product is unique since all the elements of order 2 in $\operatorname{Aut} E_8 \cong \mathrm{GL}_3(\mathbb{F}_2)$ are conjugate to each other.)

Hence the smallest n is 16.